Research Paper
Impact Factor: 4.164
Refereed Journal

IJMDRR
E- ISSN –2395-1885
ISSN -2395-1877

MAXIMAL NORMAL PRODUCT OF TWO FUZZY GRAPHS

G. Sanjeevi* **Dr. Muthukumaran****
**Associate Professor, Department of Maths, Vivekananda College, Madurai.*
***Associate Professor, Department of Maths, Saraswathi Narayanan College, Madurai.*

Abstract
In this paper we define the maximal normal product of two fuzzy graphs. We discuss the strong and complete nature of the maximal normal product of two fuzzy graphs.

Keywords: Fuzzy graphs, maximal normal product of two fuzzy graphs, strong fuzzy graphs, complete fuzzy graphs and semi complete fuzzy graphs.

1. Introduction

Fuzzy graph theory was introduced by Azriel Rosenfeld in 1975. Rosenfeld [3] has obtained the fuzzy analogues of several basic graph theoretic concepts like bridges, paths, cycles, trees and connectedness and established some of their properties. Later on Bhatacharya [11] gave some remarks on fuzzy graphs and established some connectivity concepts regarding fuzzy cut nodes and fuzzy bridges. Bhutani and Rosenfield have worked on strong arcs in fuzzy graphs [10]. Nagoorgani. A and Latha. A has worked on irregular fuzzy graphs[12]. Also Nagoorgani. A and Ratha. K has worked on regular properties of fuzzy graphs. The operations of union, join, Cartesian product and composition of two fuzzy graphs were defined by Mordeson. J.N and Peng C.S [6]. The order and size of fuzzy graphs were defined by Nagoorgani A and Basher Ahamed. B [13]. K.Radha and S. Arumugam defined the maximal products of two fuzzy graphs [14]. Here we define maximal normal product of two fuzzy graphs and we discuss the strong and complete nature of the maximal normal product of two fuzzy graphs.

2. Preliminaries

A **fuzzy graph G** is a pair of function (σ,μ) where σ is a fuzzy subset of a non-empty set V and μ is a symmetric fuzzy relation onσ. The underlying crisp graph of G: (σ,μ) is denoted by G*(V, E) where $E \subseteq V \times V$, G: (σ, μ) is called **connected fuzzy graph** if for all u, v $\in$ V there exists at least one nonzero path between u and v. G is called **strong fuzzy graph** if $\mu(u, v) = \sigma(u) \wedge \sigma(v)$ for all $(u, v) \in E$ and **complete fuzzy graph** of $\mu(u, v) = \sigma(u) \wedge \sigma(v)$, $\forall$ u, v $\in$ V. The **degree of a vertex** is of G: (σ,μ) is defined as $d_G(u) = \sum_{uv\in E} \mu(u\ v)$. The **order of a fuzzy graph** G: (σ, μ) is defined as $O(G) = \sum_{u\in V} \pm(u)$. The **size of fuzzy graph** $G(\sigma, \mu)$ is defined as $q(G) = \sum_{uv\in E} \mu(u,v)$. A **homomorphism** of fuzzy graphs G: (σ, μ) andG': (σ', μ') with underlying crisp graphs G^* (V, E) and $G^{*\prime}$ (V', E') respectively is a bijective maph: $V \rightarrow V'$ which satisfies. $\sigma(x) \leq \sigma'(h(x))$,$\forall x \in V$ and $\mu(x, y) \leq \mu'(h(x), h(y))$, $\forall x, y \in V$. A **weak isomorphism** from G to G' is a map h: $V \rightarrow V'$ which bijective homomorphism that satisfies $\sigma(x) = \sigma'(h(x))$, $\forall x \in V$.
.

3. Definition and example

Definition 3.1

Let G_1: (σ_1, μ_1) and G_2: (σ_2, μ_2) denote two fuzzy graphs with underlying crisp graphs G_1^*: (V_1, E_1) and G_2^*: (V_2, E_2) respectively. Define G: (σ, μ) where $\sigma = \sigma_1 \hat{X} \sigma_2$ and $\mu = \mu_1 \hat{X} \mu_2$ with underlying crisp graphs G: (V, E) where $V = V_1 \times V_2$ and $E = \{(u_1, u_2)(v_1, v_2) / u_1 = v_1, u_2v_2 \in E_2 \text{ or } u_2 = v_2, u_1v_1 \in E_1 \text{ or } u_1v_1 \in E_1 \text{ and } u_2v_2 \in E_2\}$ such that

$(\sigma_1 \hat{X} \sigma_2)(u, v) = \sigma_1(u) \vee \sigma_2(v)$, $\forall u \in V_1$ and $v \in V_2$and

$(\mu_1 \hat{X} \mu_2)((u_1, u_2)(v_1, v_2))$

$$= \begin{cases} \sigma_1(\mu_1) \vee \mu_2(u_2v_2), & \text{if } u_1 = v_1, u_2v_2 \in E_2 \\ \mu_1(u_1, v_1) \vee \sigma_2(u_2), & \text{if } u_1v_1 \in E_1, u_2 = v_2 \\ \mu_1(u_1v_1) \vee \mu_2(u_2v_2), & \text{if } u_1v_1 \in E_1 \text{ and } u_2v_2 \in E_2 \end{cases}$$

Then $G_1 \hat{X} G_2$: $(\sigma_1 \hat{X} \sigma_2, \mu_1 \hat{X} \mu_2)$ is called **maximal normal product** of two fuzzy graphs G_1 and G_2.

IJMDRR
E- ISSN –2395-1885
ISSN -2395-1877

Theorem 3.2

Let G_1: (σ_1, μ_1) and G_2: (σ_2, μ_2) be two fuzzy graphs with underlying crisp graphs G_1^* (V_1, E_1) and G_2^* (V_2, E_2) respectively, then their maximal normal product G: $(\sigma_1 \hat{X} \sigma_2, \mu_1 \hat{X} \mu_2)$ is a fuzzy graph.

Proof

Given that G_1: (σ_1, μ_1) and G_2: (σ_2, μ_2) are two fuzzy graphs

By the definition of maximal normal product of two fuzzy graphs, the vertex set of $G_1 \hat{X} G_2$ is $(\sigma_1 \hat{X} \sigma_2)$ (u, v) = σ_1 (u) $\vee\sigma_2$ (v), $\forall$ u $\in V_1$ and v $\in V_2$ the edge set can be found in the following three cases.

Case (1): If $u_1 = v_1$ and $u_2 v_2 \in E_2$ then

$(\mu_1 \hat{X} \mu_2)$ (u_1, u_2) (v_1, v_2) = $\sigma_1 (u_1) \vee \mu_2 (u_2v_2)$

$\leq \sigma_1(u_1) \vee [\sigma_2(u_2) \wedge \sigma_2 (v_2)]$

$= [\sigma_1 (u_1) \vee \sigma_2(u_2)] \wedge [\sigma_1(u_1) \vee \sigma_2(v_2)]$

$= [\sigma_1(u_1) \vee \sigma_2(u_2)] \wedge [\sigma_1(v_1) \vee \sigma_2(v_2)]$

$= (\sigma_1 \hat{X} \sigma_2) (u_1, u_2) \wedge (\sigma_1 \hat{X} \sigma_2) (v_1, v_2)$

$\therefore (\mu_1 \hat{X} \mu_2) (u_1, u_2) (v_1, v_2)) \leq (\sigma_1 \hat{X} \sigma_2) (u_1u_2) \wedge (\sigma_1 \hat{X} \sigma_2) (v_1v_2)$

Case (2): If $u_1v_1 \in E_1$ and $u_2 = v_2$

$(\mu_1 \hat{X} \mu_2) ((u_1, u_2) (v_1, v_2))$

$= \mu_1(u_1v_1) \vee \sigma_2(u_2)$

$\leq [\sigma_1(u_1) \wedge \sigma_1(v_1)] \vee \sigma_2(v_2)$

$=[\sigma_1 (u_1) \vee \sigma_2 (u_2)] \wedge [\sigma_1(v_1) V \sigma_2 (v_2)]$

$= (\sigma_1 \hat{X} \sigma_2) (u_1, u_2) \wedge (\sigma_1 \hat{X} \sigma_2) (v_1, v_2)$

$\therefore (\mu_1 \hat{X} \mu_2) ((u_1, u_2) (v_1, v_2))$

$\leq (\sigma_1 \hat{X} \sigma_2) (u_1, u_2) \wedge (\sigma_1 \hat{X} \sigma_2) (v_1, v_2)$

Case (3): If $u_1v_1 \in E_1$ and $u_2v_2 \in E_2$

$(\mu_1 \hat{X} \mu_2) ((u_1, u_2) (v_1, v_2))$

$= \mu_1 (u_1v_1) \vee \mu_2 (u_2, v_2)$

$\leq [\sigma_1 (u_1) \wedge \sigma_1(v_1)] \vee [\sigma_2(u_2) \wedge \sigma_2(v_2)]$

Subcase (3a): If $\sigma_1(u_1) \leq \sigma_1(v_1)$

$(\mu_1 \hat{X} \mu_2) ((u_1, u_2) (v_1, v_2))$

$\leq \sigma_1(u_1) \vee [\sigma_2 (u_2) \wedge \sigma_2(v_2)]$

$= [\sigma_1 (u_1) \vee \sigma_2(u_2)] \wedge [\sigma_1(u_1) \vee \sigma_2(v_2)]$

$\leq [\sigma_1 (u_1) \vee \sigma_2(u_2)] \wedge [\sigma_1(v_1) \vee \sigma_2(v_2)]$

$= (\sigma_1 \hat{X} \sigma_2) (u_1, u_2) \wedge (\sigma_1 \hat{X} \sigma_2) (v_1, v_2)$

$\therefore (\mu_1\mu_2) ((u_1, u_2) (v_1, v_2)) \leq (\sigma_1 \hat{X} \sigma_2) (u_1, u_2) \wedge (\sigma_1 \text{ x } \sigma_2) (v_1, v_2)$

Subcase (3b): If $\sigma_1 (v_1) \leq \sigma_1(u_1)$

$(\mu_1 \hat{X} \mu_2) ((u_1, u_2) (v_1, v_2))$

$= \sigma_1(v_1) \vee [\sigma_2(u_2) \wedge \sigma_2(v_2)]$

$= [\sigma_1(v_1) \vee \sigma_2(u_2)] \wedge [\sigma_1(v_1) \vee \sigma_2(v_2)]$

$\leq [\sigma_1(u_1) \vee \sigma_2(u_2)] \wedge [\sigma_1(v_1) \vee \sigma_2 v_2)]$

$= (\sigma_1 \hat{X} \sigma_2) (u_1, u_2) \wedge (\sigma_1 \hat{X} \sigma_2) (v_1, v_2)$

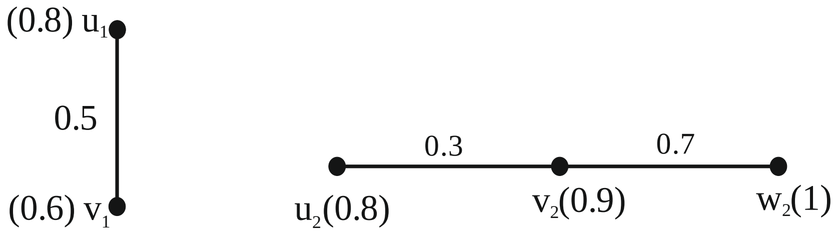
$\therefore (\mu_1 \hat{X} \mu_2) ((u_1, u_2), (v_1, v_2)) \leq (\sigma_1 \hat{X} \sigma_2) (u_1, u_2) \wedge (\sigma_1 \hat{X} \sigma_2) (v_1, v_2)$

From all the above three cases, we conclude that G: $(\sigma_1 \hat{X} \sigma_2, \mu_1 \hat{X} \mu_2)$ is a fuzzy graph.

The following example illustrates the maximal normal product of two fuzzy graphs.

Example 3.3

G_1: (σ_1, μ_2) and G_2: (σ_2, μ_2) be two fuzzy graphs and $G_1 \hat{X} G_2$:($\sigma_1 \hat{X} \sigma_2$, $\mu_1 \hat{X} \mu_2$) be their maximal normal product.



G_1: (±$_1$, «$_1$) **G_2: (±$_2$, «$_2$)**

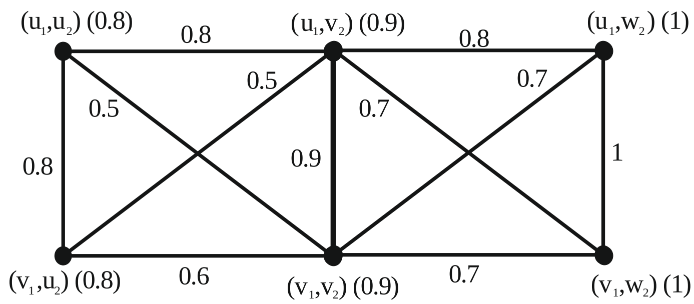
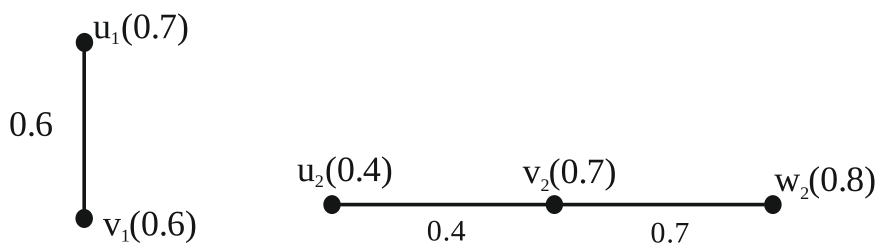


Figure 1: $G_1 \hat{X} G_2$: (±$_1 \hat{X}$ ±$_2$, «$_1 \hat{X}$ «$_2$)

The following example illustrates that maximal normal product of two strong fuzzy graphs need not be a strong fuzzy graph.

Example 3.4



G_1: (±$_1$, «$_2$) **G_2: (±$_2$, «$_2$)**
Strong fuzzy graph **Strong fuzzy graph**

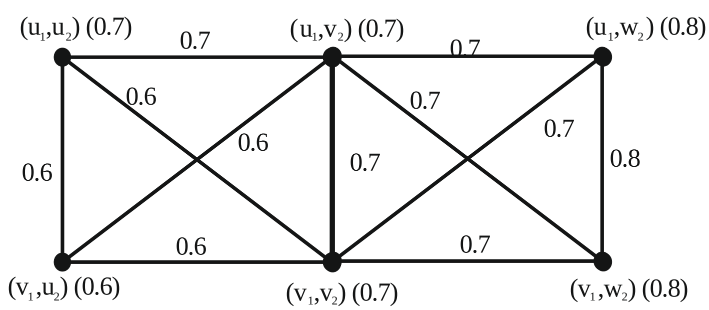


Figure 2:$G_1 \hat{X} G_2$: (±$_1 \hat{X}$ ±$_2$, «$_1 \hat{X}$ «$_2$), Not a strong fuzzy graph

In the above example we note that $(u_1, u_2)\,(v_1, v_2)$ is not an effective edge in $G_1 \hat{X} G_2$.

$\therefore\ G_1 \hat{X} G_2$ is not a strong fuzzy graph.

The maximal normal product of two complete fuzzy graphs need not be complete. So we include some conditions to make the maximal normal product to be a complete fuzzy graph. This can be seen from the following theorem.

Lemma 3.5
Let $G_1: (\sigma_1, \mu_1)$ be any fuzzy graph and $G_2: (\sigma_2, \mu_2)$ be a complete fuzzy graph such that $\sigma_1 \leq \sigma_2$ then $\mu_1\,(u_1, v_1) \leq \mu_2\,(u_2, v_2), u_1, v_1 \in V_1, u_2, v_2 \in V_2$

Proof
Since $\sigma_1 \leq \sigma_2$
$\sigma_1\,(u_1) \leq \sigma_2\,(u_2)$
and $\sigma_1\,(v_1) \leq \sigma_2\,(v_2)$

Now
$\mu_1\,(u_1, v_1) \leq \sigma_1\,(u_1) \wedge \sigma_1(v_1)$
$\leq \sigma_2\,(u_2) \wedge \sigma_2(v_2)$
$= \mu_2\,(u_2, v_2)$ ($\because G_2$ is complete)
$\therefore \mu_1\,(u_1, v_1) \leq \mu_2\,(u_2, v_2)$

Theorem 3.6
Let $G_1: (\sigma_1, \mu_1)$ and $G_2: (\sigma_2, \mu_2)$ be the complete fuzzy graphs such that $\sigma_1 \leq \sigma_2$. Then $G_1 \hat{X} G_2$ is a complete fuzzy graph.

Proof
Case (1): If $u_1 = v_1$ and $u_2v_2 \in E_2$ then
$(\mu_1 \hat{X} \mu_2)\,((u_1, u_2)\,(v_1, v_2)) = \sigma_1\,(u_1) \vee \mu_2(u_2, v_2)$
$= \sigma_1\,(u_1) \vee [\sigma_2(u_2) \wedge \sigma_2\,(v_2)]$ [$\because G_2$ is complete]
$= [\sigma_1\,(u_1) \vee \sigma_2(u_2)] \wedge [\sigma_1(u_1) \vee \sigma_2(v_2)]$
$= (\sigma_1 \hat{X} \sigma_2)\,(u_1, u_2) \wedge (\sigma_1 \hat{X} \sigma_2)\,(v_1, v_2)$

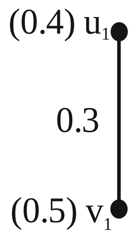
Case (2): If $u_2 = v_2$ and $u_1v_1 \in E_1$ then proceeding as in the case (1), we get
$(\mu_1 \hat{X} \mu_2)\,((u_1, u_2)\,(v_1, v_2)) = (\sigma_1 \hat{X} \sigma_2)\,(u_1, u_2) \wedge (\sigma_1 \hat{X} \sigma_2)\,(v_1, v_2)$

Case (3): If $u_1v_1 \in E_1$ and $u_2v_2 \in E_2$ then
$(\mu_1 \hat{X} \mu_2)\,((u_1, u_2), (v_1, v_2)) = \mu_1\,(u_1, v_1) \vee \mu_2\,(u_2, v_2)$
$= \mu_2\,(u_2, v_2)$ [by Lemma (5)]
$= \sigma_2(u_2) \wedge \sigma_2(v_2)$
$= [\sigma_1\,(u_1) \vee \sigma_2(u_2)] \wedge [\sigma_1\,(v_1) \vee \sigma_2(v_2)]$ [$\because \sigma_1 \leq \sigma_2$]
$= (\sigma_1 \hat{X} \sigma_2)\,(u_1, u_2) \wedge (\sigma_1 \hat{X} \sigma_2)\,(v_1, v_2)$
Hence from all the above three cases, we have
$(\mu_1\mu_2)\,((u_1, u_2)\,(v_1, v_2)) = (\sigma_1 \hat{X} \sigma_2)\,(u_1, u_2) \wedge (\sigma_1 \hat{X} \sigma_2)\,(v_1, v_2)$
$\therefore\ G_1 \hat{X} G_2$ is a complete fuzzy graph.

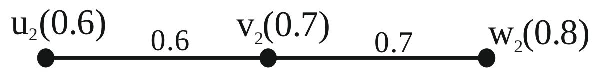
Example 3.7
Consider the following figure (3). $G_1: (\sigma_1, \mu_1)$ is an arbitrary fuzzy graph and

$G_2: (\sigma_2, \mu_2)$ is a complete fuzzy graph such that $\sigma_1 \leq \sigma_2$. Then their maximal normal product is a complete fuzzy graph.

IJMDRR
E- ISSN –2395-1885
ISSN -2395-1877



G$_1$: (±$_2$, «$_2$)
Fuzzy graph



G$_2$: (±$_2$, «$_2$)
Complete fuzzy graph

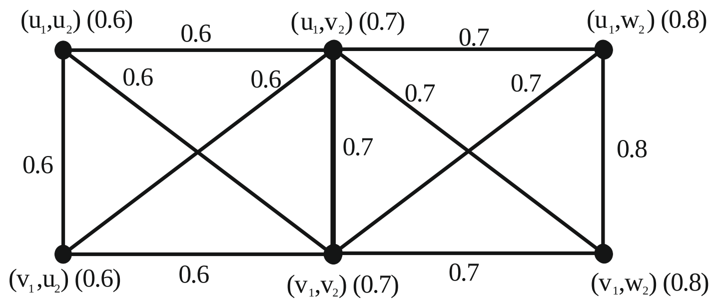
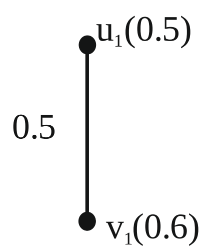


Figure 3: G$_1$ $\hat{X}$ G$_2$: (±$_1$ $\hat{X}$ ±$_2$, «$_1$ $\hat{X}$ «$_2$)

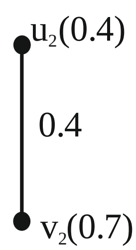
Complete fuzzy graph
In the above figure, we note that $\sigma_1 \leq \sigma_2$.

If G_1: (σ_1, μ_1) and G_2: (σ_2, μ_2) are two complete fuzzy graphs then $G_1 \hat{X} G_2$: ($\sigma_1 \hat{X} \sigma_2$, $\mu_1 \hat{X} \mu_2$) is semi complete fuzzy graph.

Example3.8



G$_1$: (±$_1$, «$_1$)
Complete fuzzy graph



G$_2$: (±$_2$, «$_2$)
Complete fuzzy graph

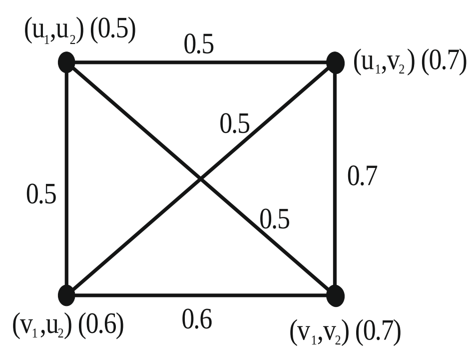


Figure 4: G$_1$ $\hat{X}$ G$_2$: (±$_1$ $\hat{X}$ ±$_2$, «$_1$ $\hat{X}$ «$_2$)

Semi Complete Fuzzy Graph

In the above figure we note that $(G \hat{X} G_2)^*$ is a complete graph.

Therefore $G_1 \hat{X} G_2$ is a semi complete fuzzy graph

4. Conclusion

We define the maximal normal product of two fuzzy graphs.We have given illustrationfor the maximal normal product of two fuzzy graphs. By giving an example we illustrates that maximal normal product of two strong fuzzy graphs need not be a strong fuzzy graph. We have proved that under some condition $G_1 \hat{X} G_2$ is a complete fuzzy graph.

References

1. L.A.Zadeh, Fuzzy sets, "Information and control 8, 1965, P. 338 – 353.
2. A. Kaufman, "Theory of Fuzzy Subset", Vol.1, Academic Press, New York, 1975.
3. Rosenfeld A., Fuzzy graphs In: L.A.Zadeh., K.S.Fu. M. Shimura (Eds), "Fuzzy sets and their applications to cognitive and decision process", Academic Press, New York, 1975, P. 77-95.
4. John N. Mordeson and Premchand S. Nair, Fuzzy Graphs and Fuzzy Hypergraphs, Physica – Verlag, Heidelberg, 2000, New York.
5. P.S.Nair, Traingle and parallelogram laws on fuzzy graphs, Pattern Recognition letters 15, 1994, P. 803-805.
6. John N. Mordeson and Chang – Shyh Peng, "Operations on fuzzy graphs, Information Sciences 79, 1994, P. 159-170.
7. Mordeson J.N., "Fuzzy line graphs", Pattern Recognition Letters, 14, 1993, P. 381-384.
8. Kiran R. Bhutani and Azriel Rosenfeld, "Fuzzy end notes in fuzzy graph", Information Sciences '152, 2003, P. 323-326.
9. Prabhir Bhattacharya, "Fuzzy relations and fuzzy groups," Information Sciences 1985, P 267-282.
10. Kiran R. Bhutani and Azriel Rosenfeld, "Strong arcs in fuzzy graphs", Information Sciences 152, 2003, P 319-322.
11. Bhattacharya, P. "Some remarks on fuzzy graph", Pattern Recognition Letters 6, 1987, P. 297-302.
12. Nagoorgani. A and Latha. A "On irregular fuzzy graphs", Applied Mathematical Sciences, 6(11), 517-523, 2012.
13. Nagoor Gani, A., and Basheer Ahamed, M, "Order and Size in Fuzzy Graphs," Bulletin of Pure and Applied Sciences, 22E(No 1), (2003), 145-148.
14. Radha. K and S. Arumurgam "On maximal product of two fuzzy graphs" International Journal of Current Research, vol.7, Issue, pp.11508-111515, January 2015.