Research Paper
Impact Factor: 4.164
Refereed Journal

IJMDRR
E- ISSN –2395-1885
ISSN -2395-1877

DECOMPOSITION OF CURVATURE TENSOR FIELDS IN A FINSLER SPACE

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Introduction

K. Yano [5]Defined the normal-projective curvature tensor as under:

(1.1) $N^i_{jkh} = \partial_h \Pi^i_{kj} + \Pi^i_{lkj}\Pi^l_{kp}x^p + \Pi^i_{lh}\Pi^l_{kj}$(-k/h)

where

(1.2) $\Pi^i_{jkh} = \partial_j \Pi^i_{kh} = G^i_{jkh} - \frac{1}{n+1}\left(\delta^i_j G^r_{khr} + x^i G^r_{jkhr}\right)$

and

(1.3) $G^i_{ljkh} = \partial_l G^i_{jkh}$

Here Π^i_{jkh} constitute the components of a tensor and Yano [   ] denoted this tensor by U^i_{jkh}. We shall abide by the notations as has been suggested by Yano and shall denote the tensor $\Pi^i_{jkh}$by U^i_{jkh} from here onwards. Thus,

(1.4) $U^i_{jkh} = G^i_{jkh} - \frac{1}{n+1}\left(\delta^i_j G^r_{khr} + x^i G^r_{jkhr}\right)$

This tensor satisfies the following identities and contractions

(1.5) (a) $U^i_{jkh} = U^i_{jhk}$,   (b) $U^i_{jki} = G^i_{jki}$

(c) $U^i_{jkh}x^j = 0$,   (d) $U^i_{jkh}x^h = \frac{1}{n+1}x^i G^h_{jkh}$,

(e) $U^i_{ikh} = \frac{2}{n+1}G^i_{ikh}$

A relation in between the normal projective curvature tensor and The Berwald curvature tensor has been obtained by P.N. Pande [i] and it is given by

(1.6) $N^i_{jkh} = H^i_{jkh} - \frac{1}{n+1}x^i \partial_j H^r_{rkh}$

Contracting the indices i and j in (1.6) and Thereafter using the fact that H^r_{rkh} is positively homogenous of degree zero in x^i, we get

(1.7) $N^r_{rkh} = H^r_{rkh}$

Transvection of (1.6) by x^i gives

(1.8) $N^i_{jkh}x^i = H^i_{kh}$,

Where we have taken into account the fact that

$H^i_{hjk}x^h = H^i_{jk}$

A connection in between the normal projective curvature tensor N^i_{jkh} and the projective curvature tensor W^i_{jkh} has been obtained in the following form:

(1.9) $W^i_{jkh} = N^i_{jkh} + \left(\delta^i_j M_{hj} - \delta^i_j M_{kh}\right)$ $(-k/h)$

where

(1.10) $M_{kh} = -\frac{1}{n^2-1}(nN_{kh} + N_{hk})$

and
(1.11) $N_{jk} = N^r_{jkr}$

If can easily be verified that the normal projective curvature tensor is skew symmetric in its last two indices, i.e.
(1.12) $N^i_{jkh} = -N^i_{jkh}$

It has also been seen that the normal projective curvature tensor satisfies the identity
(1.13) $N^i_{jkh} + N^i_{khj} + N^i_{hjk} = 0$

Contracting (1.6) with respect to the indices i and h, we get
(1.14) $N_{jk} = H_{jk} - \frac{1}{n+1} x^i \partial_j H^r_{rki}$

An equivalent alternative from of (1.14) is given as
(1.15) $N_{jk} = H_{jk} - \frac{1}{n+1}\{\partial_j(H^r_{rki}x^i) - H^r_{rkj}\}$

Another alternative form of (1.15) can also be given as
(1.16) $N_{jk} = \frac{n}{n+1} H_{jk} - \frac{1}{n+1} H_{kj} + \frac{n-1}{n+1}\partial_j\partial_k H$
(1.16) gives a relationship in between two Ricci tensors N_{jk} and H_{jk}

Decomposition of Normal Projective Curvature Tensor
The normal projective curvature tensor N^i_{jkh} is a mixed tensor of order 4 with contra variant valancy 1 and covariant valancy 3 hence the most expected forms of decomposition of this tensor may be given as under :-
(2.1) $N^i_{jkh} = X^i Y_{jkh},$
(2.2) $N^i_{jkh} = X_j Y^i_{kh},$
(2.3) $N^i_{jkh} = X_k Y^i_{jh},$
(2.4) $N^i_{jkh} = X_h Y^i_{jk},$
(2.5) $N^i_{jkh} = X^i_k Y_{jh},$
(2.6) $N^i_{jkh} = X^i_h Y_{jk},$
(2.7) $N^i_{jkh} = X^i_j Y_{kh},$

We propose to discuss these possibilities one by one.
We now consider a Finsler space whose normal projective curvature tensor is of the form (2.1). Transvecting (2.1) by and thereafter using (1.6) we get:
(2.8) $H^i_{kh} = X^i Y_{jhk} x^t$

We newtransvect (2.8) by y^i thereafter and observe that at least one of the following two conditions always holds
(2.9) $(a) y_i X^i = 0, \quad (b) Y_{jkh} x^t = 0$

if (2.9a) is time then transvecting (2.1) by y_i, we get
(2.10) $y_i N^i_{jkh} = 0$

Using (1.6) is (2.10), we get,
(2.11) $y_i H^i_{jkh} = \frac{1}{n+1} F^2 \partial_j H^r_{rkh}$

IJMDRR
E- ISSN –2395-1885
ISSN -2395-1877

Using (1.6) is (2.11), we get,

(2.12) $N^i_{jkh} = H^i_{jkh} - l^i l_r H^r_{jkh}$

(2.12) can alternatively be written in the following form

(2.13) $N^i_{jkh} = h^i_r H^r_{jkh}$

Where

(2.14) $h^i_r = \partial^i_r - l^i l_r$

These observations clearly tell that in (2.1) the vector X^i cannot be independent of x^i for otherwise $y_i X^i = 0$ which will lead to $X^i = 0$ and alternatively $N^i_{jkh} = 0$. Similarly the condition (3.96) will also not hold because if will load to $H^i_{kh} = 0$ which will imply $H^i_{jkh} = 0$ and hence $N^i_{jkh} = 0$. Therefore, from this observation, we may state:

Theorem (2.1)

If we assume the decomposition of the normal projective curvature tensor N^i_{jkh} of a Finsler space in the form (2.1) and assume that the decompose vector field X^i is not independent of directional argument then the normal projective curvature tensor and the Berwald's curvature tensor of the Finsler space are connected by (2.13).

In addition to all these conditions, if we assume that the space under consideration is of recurrent curvature i.e. $H^i_{jkh(m)} = \lambda_m H^i_{jkh}$. Then it can easily be verified that the space under consideration is normal projective recurrent. Therefore, we can state:

Theorem (2.2)

If the recurrent normal projective curvature tensor of a Finsler space be decomposable in the form (2.1) then the space under consideration is necessarily a normal projective recurrent Finsler space.

Now, we consider a Finsler space in which the normal projective curvature tensor field is decomposable in the form (2.2). Transvecting (2.2) by x^j and using (1.8), we get

(2.15) $H^i_{kh} = X_j x^t Y^i_{kh}$

At least one of the following two conditions will always hold, if we transvect (2.15) by y_i

(2.16) $(a) X_j x^t = 0, \quad (b) y_i Y^i_{kh} = 0$

If we assume (2.16a) to be true then from (2.15) we get $H^i_{kh} = 0$which will obviously imply $H^i_{jkh} = 0$and this implication will also load to $N^i_{jkh} = 0$.Hence, we find from here that (2.16a) cannot hold. Hence (2.16b) will hold. Transvection of (2.2) by , after making use of (2.16b) gives –

(2.17) $y_i N^i_{jkh} = 0$

With the help of (2.17) and (1.6), we get

(2.18) $y_i H^i_{jkh} = \frac{1}{n+1} F^2 \partial_j H^r_{rkh}$

Using (2.18) in (1.6), we get

(2.19) $N^i_{jkh} = H^i_{jkh} - l^i l_r H^r_{jkh} = h^i_r H^r_{jkh}$

where we have taken into account (2.14), therefore, we can state:

Theorem (2.3)

If the normal projective curvature tensor N^i_{jkh} of a Finsler space be supposed to be (i) decomposable in the form (2.2) and (ii) the decompose vector field be not independent of directional arguments then the normal projective curvature tensor and the Berwald's curvature tensor of the Finsler space are connected by a relation of the form (2.19).

Like the previous supposition here also we suppose that the Finsler space under consideration is of recurrent curvature, i.e. $H^i_{jkh(m)} = \lambda_m H^i_{jkh}$ then it can easily be verified that the space under consideration is normal projective recurrent. Therefore, we can state:

Theorem (2.4)
If the normal projective curvature tensor of a Finsler space be of recurrent curvature then the space under consideration is normal projective recurrent.
We now consider the case in which the normal projective curvature tensor is decomposable in the form (2.3). Transvecting (2.3) by x^t and using (1.8), we get

(2.20) $H^i_{kh} = X_k x^t Y^i_{jh}$

Multiplying (2.20) by X_m and thereafter taking skew-symmetric part with respect to the indices k and m, we get,

(2.21) $X_m H^i_{kh} = X_k H^i_{mh}$

Transvecting (2.21) by x^m and thereafter using the fact that $H^i_{jk} x^t = H^i_k$, we get

(2.22) $X_m x^m H^i_{kh} = X_k H^i_h$

Transvecting (2.22) by x^h and then using the fact that $H^i_k x^k = 0$, we get

(2.23) $X_m x^m H^i_{kh} = 0$

(2.23) enables usto state that atleast one of the following two conditions will always hold:

(2.24) $(a) X_m x^m = 0, \quad (b) H^i_k = 0$

Using (2.24a) in (2.22) we have $X_k H^i_h = 0$ which imply $X_k = 0$or $H^i_h = 0$, the condition $X_k = 0$tells that $N^i_{jkh} = 0$hence $X_k = 0$is net possible.

Therefore, only one alternative left is $H^i_h = 0$ which also will not hold because it will automatically imply $N^i_{jkh} = 0$. Thus, we fuid that either of the two conditions given by (2.24) does not hold in case we consider the decomposition of the normal projective curvature tensor in the form (2.3), Therefore, we can state:

Theorem (2.5)
The normal projective curvature tensor $N^i_{jkh}$of a Finsler space cannot be decomposed in the form (2.3).

We now consider a Finsler space where normal projective curvature tensor is decomposable in the form (2.4). Transvecting (2.4) by x^t and thereafter using (1.8), we get

(2.25) $H^i_{kh} = x^t X_h Y^i_{jk}$

Multiplying (2.25) by X_m and taking skew-symmetric part with respect to the indices h and m, we get,

(2.26) $X_m H^i_{kh} = X_h H^i_{km}$

Transvecting (2.26) byx^m, we get

(2.27) $x^m X_m H^i_{kh} = -X_h H^i_k$

Transvecting (2.27) by x^k, we get

(2.28) $X_m x^m H^i_h = 0$

The above equation implies that atleast one of the following two conditions will always hold:

(2.29) $(a) X_m x^m = 0, \quad (b) H^i_h = 0$

Using (2.29a) in (2.27) we get,

(2.30) $X_h H^i_h = 0$

(2.30) obviously implies that either $X_h = 0$or $H^i_k = 0$. The condition $X_h = 0$with tell$N^i_{jkh} = 0$ henc, we conclude that $X_h = 0$is not possible i.e.$X_h \neq 0$. The condition $H^i_k = 0$is also not possible because it two leads to$N^i_{jkh} = 0$. Thus, these observations enable us to state that condition (2.29a) is not possible. Similarly it can be seen that condition (2.29b) is also not possible. Therefore, we can state:

Theorem (2.6)
The normal projective curvature tensor of a Finsler space is not be decomposed in the form (2.4).
We now consider a Finsler space in which the normal projective curvature tensor is decomposable in the form (2.5). Transvecting (2.5) by x^j and thereafter using (1.8), we get

(2.31) $H^i_{kh} = X^i_k x^j Y_{jh}$

Transvecting (2.31) byy_i, we find that under the decomposition (2.5) atleast one of the following two will always hold:

(2.32) $(a) X^i_k y_i = 0, \quad (b) Y_{jh} x^j = 0$

If we now assume that (2.32a) is true then transvection of (2.5) by $y_i$gives $N^i_{jkh} = 0$, this observation, in view of (1.6) will give

(2.33) $y_i H^i_{jkh} = \frac{1}{n+1} F^2 \partial_j H^r_{rkh}$

using (2.33) is (1.6), we get,

(2.34) $N^i_{jkh} = H^i_{jkh} - l^i l_r H^r_{jkh}$

In the light of (2.14), we may reusite (2.34) in the following form

(2.35) $N^i_{jkh} = h^i_r H^r_{jkh}$

The condition (2.326) will also not hold because in view of (2.31) it will lead to $H^i_{kh} = 0$ which automatically implies $H^i_{jkh} = 0$ and hence $N^i_{jkh} = 0$. Therefore, we may state:

Theorem (2.7)
If we assume that the normal projective curvature tensor $N^i_{jkh}$of a Finsler space is decomposable in the form (2.5) then the connection in between the normal projective curvature tensor and Berwald's curvature tensor of a Finsler space is given by (2.35).
If we now suppose that the space under consideration is of recurrent curvature, i.e. $H^i_{jkh(m)} = \lambda_m H^i_{jkh}$ then it can easily be verified that such a space is normal projective recurrent. Therefore, we can state:

Theorem (2.8):
If the recurrent normal projective curvature of a Finsler space be decomposable in the form (2.5) then such a space is always normal projective recurrent.
We now consider a Finsler space in which the normal projective curvature tensor is decomposable in the form (2.6). Transvecting (2.5) by x^j and thereafter using (1.8), we get

(2.36) $H^i_{kh} = X^i_k x^j Y_{jh}$

Transvecting (2.36) by y_i, we find that at least one of the following two conditions always holds:

(2.37) $(a) X^i_k y_i = 0, \quad (b) Y_{jh} x^j = 0$

If the condition (2.37a) be supposed to be true then transvecting (2.5) byy_i, we get,

(2.38) $y_i N^i_{jkh} = 0$

using (1.8) is (2.38), we get,

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E- ISSN –2395-1885
ISSN -2395-1877

(2.39) $y_i H^i_{jkh} = \frac{1}{n+1} \partial_j H^r_{rkh}$

Using (2.39) in (1.8), we may rewrite (1.8) in the following alternative form

(2.40) $N^i_{jkh} = H^i_{jkh} - l^i l_r H^r_{jkh}$

Using (2.14) in (2.40), we get

(2.41) $N^i_{jkh} = h^i_r H^r_{jkh}$

The condition (2.37b) cannot hold because in view of (2.36) this condition will lead to $H^i_{kh} = 0$ which in turn will imply $H^i_{jkh} = 0$, Therefore, we may state:

Theorem (2.9)

If we suppose that the normal projective curvature tensor of a Finsler space is decomposable in the form (2.6) then the normal projective curvature tensor and Berwald's curvature tensor of a Finsler space are connected by (2.41).

If we now consider that the space under consideration is of recurrent curvature, i.e. $H^i_{jkh(m)} = \lambda_m H^i_{jkh}$ then it can easily be verified that such a space is normal projective recurrent. Therefore, we can state:

Theorem (2.10)

If the recurrent normal projective curvature tensor of a Finsler space be assumed to be decomposable in the form (2.6) then such a space is always normal projective recurrent.

We now consider the last case in which the normal projective curvature tensor of the Finsler space is decomposable in the form (2.7). Transvecting (2.7) by x^j and thereafter using (1.8), we get

(2.42) $H^i_{kh} = X^i_k x^j Y_{jh}$

Transvecting (2.42) by y_i, we get

(2.43) $X^i_j x^j \quad y_i Y_{kh} = 0,$

Form (2.43) we conclude that either of the following two conditions will always hold the space under consideration

(2.44) $(a) X^i_k x^j y_i = 0, \quad (b) Y_{kh} = 0$

If the condition (2.44ba) automatically leads to $N^i_{jkh} = 0$, therefore such a condition is always not possible. Hence, only alternative left with us is to consider the case:

(2.45) $X^i_j x^j y_i = 0$

Contracting (2.7) with respect to the indices i and j and then using (1.9), we get

(2.46) $H^r_{rkh} = H_{hk} - H_{kh} = X^r_r Y_{kh}$

From (2.46), we have

(2.47) $Y_{kh} = \frac{1}{X}(H_{hk} - H_{kh}) \quad where \quad X = X^r_r$

Therefore, we can state:

Theorem (2.11)

If the normal projective curvature tensor of a Finsler space be assumed to be decomposable in the form (2.7) then the tensors X^i_j and Y_{kh} always satisfy (2.45) and (2.47).

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