Research Paper
Impact Factor: 4.164
Refereed Journal

IJMDRR
E- ISSN –2395-1885
ISSN -2395-1877

ON HIGHER ORDER RECURRENT FINSLER SPACES

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Introduction

The covariant derivative of any tensor field T_j^i in the sense of Cartan [1] is defined as:

(1.1) $$T^i_{j|k} = \partial_k T^i_j + T^m_j F^i_{mj} - T^i_m F^m_{jk}$$

The Ricci identity for a tensor T^i_j in the sense of Cartan [2] is given by

(1.2) $$T^i_{j|h|k} - T^i_{j|k|h} = T^m_j R^i_{mhk} - T^i_m R^m_{jhk} - T^i_{j|m} R^m_{hk},$$

Where

(1.3) $$R^i_{hkm} = -_{(km)}\left\{\frac{\partial \Gamma^{*i}_{hk}}{\partial x^m} - \frac{\partial \Gamma^{*i}_{hk}}{\partial \dot{x}^r} G^r_m + \Gamma^{*r}_{hk}\Gamma^{*i}_{rm} + C^i_{hr} R^r_{km}\right\},$$

(1.4) $$R^i_{hk} = R^i_{hkm}\dot{x}^m = -_{(hk)}\left\{\frac{\partial G^i_h}{\partial x^k} - \frac{\partial G^i_h}{\partial \dot{x}^m} G^m_k\right\}$$

And $-_{(hk)}\{\}$ denotes the interchange of the indices h and k and subtraction thereafter . The Cartan's curvature Tensor satisfies the following identities :

(1.5) $$(a)\ R_{hijk} = -R_{hikj}\ (b)\ R_{hijk} = -R_{ihjk}\ (c)\ R_{ijk} = -R_{ikj}$$

Where $R_{Gijk} = g_{mi} R^m_{hjk}$ and $R_{ijk} = g_{mi} R^m_{jk}$

The deviation tensors $H^j_k(x,\dot{x})$, $H^i_{jk}(x,\dot{x})$ and Berwald's Curvature tensor $H^i_{hjk}(x,\dot{x})$ satisfies the following :

(1.6) (a) $$H^i_{jk}(x,\dot{x}) = K^i_{hjk}(x,\dot{x})\dot{x}^h = R^i_{hjk}(x,\dot{x})\dot{x}^h,$$

(b) $$H^i_{hjk} = K^i_{hjk} + \dot{x}^r \dot{\partial}_h K^i_{rjk}.$$

And

(1.7)

(a) $$H^i_k(x,\dot{x})\dot{x}^k = 0,$$

(b) $$H^i_{jk}\dot{x}^j = H^i_k$$

(c) $$H^i_{hjk}\dot{x}^h = H^i_{jk},$$

(d) $$H^h_{ih} = H_i,$$

(e) $$H^h_{ijh} = H_{ij} = \dot{\partial}_i H_j,$$

(f) $$H_{ij}\dot{x}^i = H_j,$$

(g) $$H_i\dot{x}^i = H^i_i = (n-1)H_j,$$

(h) $$H^i_{hjk} + H^i_{jkh} + H^i_{khj} = 0,$$

(i) $$H_{jh} - H_{hj} = H^k_{khj}$$

The curvature tensor $K^i_{hjk}(x,\dot{x})$ appearing in (1.6a) is given by

(1.8) $$K^i_{jhk} = -_{(hk)}\left\{\frac{\partial \Gamma^{*i}_{jk}}{\partial x^k} - \frac{\partial \Gamma^{*i}_{jh}}{\partial \dot{x}^r} G^r_k + \Gamma^{*i}_{mk}\Gamma^{*m}_{jh}\right\}$$

And this curvature tensor satisfies the following identities

(1.9) **(a)** $K^{\,i}_{jhk} = -K^{\,i}_{jkh}$,

(b) $K^{\,i}_{jhk} + K^{\,i}_{hkj} + K^{\,i}_{kjh} = 0$,

(c) $K_{jihk} = -K_{ijh} - 2C_{ijm}K^{m}_{rhk}\dot{x}^{r}$

Where $K_{ijhk} \overset{def}{=} g_{rj}K^{\,r}_{ihk}$

R^h -Generalised Birecurrent Finsler Spaces

Verma [9] discussed a Finsler space in which Cartan's third Curvature tensor $R^{\,i}_{jkh}(x,\dot{x})$ satisfies the recurrence property with respect to Cartan's connection Γ^{*i}_{jk} and she called it as R^h -recurrent space. Thus, an R^h -recurrent space is characterised by

(2.1) $R^{\,i}_{jkh|m} = a_m R^{i}_{jkh}$,

Here, the non – zero vector fields $a_m(x)$ is called a recurrence vector field.

Dixit [3] discussed a more general Finisler space in which the Cartan's third curvature tensor satisfies the birecurrence condition with respect to condition with respect to Cartan's connection Γ^{*i}_{jk} and she called it and R^h -birecurrent space. Thus, an R^h –birecurrent spaces is characterised by

(2.2) $R^{\,i}_{jkh|m|\ell} = \sim_{\ell m} R^{i}_{jkh},\ R^{i}_{jkh} \neq 0$

Where $\sim_{\ell m}$ appearing in (2.2) is a non-zero covariant tensor field of second order and is called recurrence tensor field.

We now consider a Finsler space in which the Cartan's third curvature tensor satisfies.

(2.3) $R^{\,i}_{jkh|m|\ell} = a_\ell R^{\,i}_{jkh|m} + \sim_{\ell m} R^{i}_{jkh}$

And

(2.4) $R^{\,i}_{jkh|m|\ell} = a_m R^{\,i}_{jkh|\ell} + \sim_{\ell m} R^{\,i}_{jkh}$

Where a_l and $\sim_{\ell m}$ are non-zero covariant vector and covariant tensor field of order 2 respectively. The space satisfying (2.3) and (2.4) will respectively be called as R^h -generalized birecurrent space of first and second kinds, we shall briefly denote them by $R^h - GBRF_n - I$ and $R^h - GBRF_n - II$ respectively.

In particular, if the space satisfies.

(2.5) $R^{\,i}_{jkh|m|\ell} = a_\ell R^{\,i}_{jkh|m}$

And

(2.6) $R^{\,i}_{jkh|m|\ell} = a_m R^{\,i}_{jkh|\ell}$

Where a_r is a non – zero covariant vector field, then such a Finsler space will respectively be termed as special R^h -generalised bireccurrent space of first and second kinds and shall briefly be denoted by $R^h - SGBRF_n I$ and $R^h - SGBRF_n II$ respectively. If we consider a_ℓ as zero in (2.3) then (2.3) immediately reduces into (2.2) which will be the condition to be satisfied by the curvature tensor in a birecurrent space. Thus, we conclude that $a_\ell = 0$ reduces an R^h -generalised birecurrent space of the first kind into an R^h -birecurrent space and similarly the assumption $a_m = 0$ too will reduce an R^h -generalised binecurrent space of the second kind into an R^h -birecurrent space. Transecting (2.3), (2.4), (2.5) and (2.6) respectively by g_{ip}, we get

IJMDRR
E- ISSN –2395-1885
ISSN -2395-1877

(2.7) $R_{jpkh|m|\ell} = \text{a}_{\ell} R_{jpkh|m} + \sim_{\ell m} R_{jpkh}$,

(2.8) $R_{jpkh|m|\ell} = \text{a}_{m} R_{jpkh|\ell} + \sim_{\ell m} R_{jpkh}$,

(2.9) $R_{jpkh|m|\ell} = \text{a}_{l} R_{jpkh|m}$

and

(2.10) $R_{jpkh|m|\ell} = \text{a}_{m} R_{jpkh|m}$

Where, we have taken into account the fact that the metric tensor g_{ij} of a Finsler space is a covariant constant. Conversely, if we transvect (2.7), (2.8), (2.9) and (2.10) by g^{ip} we immediately get (2.3), (2.4), (2.5) and (2.6) respectively. Therefore, we can state:

Theorem (2.1)
In a R^h -generalised birecurrent and in an special R^h -generalised birecurrent Finsler spaces of the two kinds the conditions (2.3), (2.4), (2.5) and (2.6) are respectively equivalent to (2.7), (2.8), (2.9) and (2.10).

Theorem (2.2)
R^h -generalised birecurrent and special R^h -generalised birecurrent spaces of the two kinds may respectively be characterised by the conditions (2.7), (2.8), (2.9) and (2.10).
We now contract (2.3), (2.4), (2.5) and (2.6) with respect to the indices i and h and get

(2.11) $R_{jk|m|\ell} = \text{a}_{\ell} R_{jk|m} + \sim_{\ell m} R_{jk}$,

(2.12) $R_{jk|m|\ell} = \text{a}_{m} R_{jk|\ell} + \sim_{\ell m} R_{jk}$,

(2.13) $R_{jk|m|\ell} = \text{a}_{\ell} R_{jk|m}$

and

(2.14) $R_{jk|m|\ell} = \text{a}_{m} R_{jk|\ell}$

Respectively, therefore, we can state:

Theorem (2.3)
The Ricci tensor R_{jk} of generalised birecurrent and R^h -special generalised birecurrent Finsler spaces of the two kinds respectively satisfy (2.11),(2.12), (2.13) and (2.14).

However, if the Ricci tensor of a Finsler space satisfy (2.11) or (2.12) then it can be seen that such tensor need not be R^h - generalised birecurrent of the first kind or R^h -generalised birecurrent of the second kind similarly if the Ricci tensor of a Finsler space satisfies (2.13) or (2.14) then it can also be seen that such a Finsler space need not be R^h -special generalised birecurrent of the first kind or R^h -generalised birecurrent of the second kind. We shall now investigate the circumstances under which this holds, the curvature tensor R_{ijkh} of a three dimensional Finsler space is given in the form [4]

(2.15) $R_{ijkh} = g_{ik} L_{jh} + g_{jh} L_{ik} \left(-i/h\right)$

Where

(2.16) (a) $$L_{ik} = \frac{1}{n-2}\left(R_{ik} - \frac{\text{¥}}{2} g_{ik}\right)$$

(b) $$\text{¥} = \frac{1}{n-1} R_i^i .$$

Transvecting (2.11), (2.12), (2.13) and (2.14) respectively by g^{jp} , we get

(2.17) $R^p_{k|m|\ell} = \text{a}_{\ell} R^p_{k|m} + \sim_{\ell m} R^p_k$,

(2.18) $R^{p}_{k|m|\ell} = a_{m} R^{p}_{k|\ell} + \sim_{\ell m} R^{p}_{k}$,

(2.19) $R^{p}_{k|m|\ell} = a_{\ell} R^{p}_{k|m}$

and

(2.20) $R^{p}_{k|m|\ell} = a_{m} R^{p}_{k|\ell}$.

Contracting (2.17), (2.18), (2.19) and (2.20) with respect to the indices p and k and thereafter using (2.16b), we get

(2.21) $\text{¥}_{|m|\ell} = a_{\ell} \text{¥}_{|m} + \sim_{\ell m} \text{¥}$,

(2.22) $\text{¥}_{|m|\ell} = a_{m} \text{¥}_{|\ell} + \sim_{\ell m} \text{¥}$,

(2.23) $\text{¥}_{|m|\ell} = a_{\ell} \text{¥}_{|m}$

and

(2.24) $\text{¥}_{|m|\ell} = a_{m} \text{¥}_{|\ell}$

In view of the above four equations (2.11), (2.12), (2.13), and (2.14), the second covariant differentiation of (2.16a) with respect to x^m in the sense of Cartan gives

(2.25) $L_{ik|m|\ell} = a_{\ell} L_{ik|m} + \sim_{\ell m} L_{ik}$,

(2.26) $L_{ik|m|\ell} = a_{m} L_{ik|\ell} + \sim_{\ell m} L_{ik}$,

(2.27) $L_{ik|m|\ell} = a_{\ell} L_{ik|m}$

and

(2.28) $L_{ik|m|\ell} = a_{m} L_{ik|\ell}$

Differentiating (2.15) covariantly twice with respect to x^{ℓ} and x^{m} successively in the sense of Cartan and using the equations (2.25), (2.26), (2.27) and (2.28) we respectively get (2.3), (2.4), (2.4) and (2.6) therefore, we can state :

Theorem (2.4)
A three dimensional Ricci generalised and Ricci special generalised binecurrent Finsler spaces of the two kinds are necessarily R^h– special generalised birecurrent of the two kinds respectively.

Matsumoto [4] introduced a Finsler space $F_n(n>3)$ for which the tensor R_{ijkh} satisfies (2.15) and he called such a space as R – 3like Finsler space. If we consider an R-3 -like Ricci generalised and Ricci special generalized spaces of the two kinds and thereafter applying the same process as have been applied in the foregoing lines, we get

(2.29) $\mathrm{R}_{ijkh|m|l} = a_{\ell} R_{ijkh|m} + \sim_{\ell m} R_{ijkh}$,

(2.30) $\mathrm{R}_{ijkh|m|l} = a_{m} R_{ijkh|\ell} + \sim_{\ell m} R_{ijkh}$,

(2.31) $\mathrm{R}_{ijkh|m|l} = a_{\ell} R_{ijkh|m}$

and

(2.32) $\mathrm{R}_{ijkh|m|l} = a_{m} R_{ijkh|\ell}$.

Therefore, we can state:

Theorem (2.5)
R^h–generalised and R^h –special generalised birecurrent spaces of the two kinds are respectively Ricci generalised and Riccispecial generalised birecurrent of the two kinds but the converse of this statement is not true. However, if the space F_n is R – 3like then the converse is also true.

Transvecting (2.3), (2.4), (2.5) and (2.6) by $\dot{x}^{j}$, we get

(2.33) $H^{i}_{kh|m|\ell} = a_{\ell} H^{i}_{kh|m} + \sim_{\ell m} H^{i}_{kh}$,

(2.34) $H^{i}_{kh|m|\ell} = a_{m} H^{i}_{kh|\ell} + \sim_{\ell m} H^{i}_{kh}$,

IJMDRR
E- ISSN –2395-1885
ISSN -2395-1877

(2.35) $H^{\;i}_{kh|m|\ell} = \text{a}_{\ell} H^{\;i}_{kh|m}$

and

(2.36) $H^{\;i}_{kh|m|\ell} = \text{a}_{m} H^{\;i}_{kh|\ell}$.

Transvecting (2.33), (2.34),(2.35) and (2.36) by , $\dot{x}^{k}$ we get

(2.37) $H^{i}_{h|m|\ell} = \text{a}_{\ell} H^{i}_{h|m} + \sim_{\ell m} H^{i}_{h}$,

(2.38) $H^{i}_{h|m|\ell} = \text{a}_{m} H^{i}_{h|\ell} + \sim_{\ell m} H^{i}_{h}$,

(2.39) $H^{\;i}_{h|m|\ell} = \text{a}_{\ell} H^{\;i}_{h|m}$

and

(2.40) $H^{\;i}_{h|m|\ell} = \text{a}_{m} H^{\;i}_{h|\ell}$

Contracting (2.33), (2.34), (2.35) and (2.36) with respect to the indices i and h,we get

(2.41) $H_{|m|\ell} = \text{a}_{\ell} H_{|m} + \sim_{\ell m} H$,

(2.42) $H_{|m|\ell} = \text{a}_{m} H_{|\ell} + \sim_{\ell m} H$,

(2.43) $H_{|m|\ell} = \text{a}_{\ell} H_{|m}$

and

(2.44) $H_{|m|\ell} = \text{a}_{m} H_{|\ell}$

Therefore, we may state:

Theorem (2.6)

The tensors $H^{\;i}_{kh}$, $H^{\;i}_{h}$, the vectorc H_{k} and the scalar H of R^{h} –generalised and R^{h} –special generalised birecurrent Finsler spaces of the two kinds are respectively h– special generalised birecurrent of the two kinds.

From here onwards we propose to establish the necessary and sufficient condition in order that the Berwald's curvature tensor $H^{\;i}_{jkh}$ may become R^{h}–generalised birecurrent of first and second kinds and also it be h –special generalised birecurrent of first and second kinds -

Differentiating (2.33) patially with respect to $\dot{x}^{j}$ and thereafter using (1.2), we get

(2.45) $$\dot{\partial}_{j} H^{\;i}_{kh|m|\ell} = \left(\dot{\partial}_{j} \text{a}_{\ell}\right) H^{\;i}_{kh|m} + \text{a}_{\ell} \dot{\partial}_{j} \left(H^{\;i}_{kh|m}\right) + \\ + \left(\dot{\partial}_{j} \sim_{\ell m}\right) H^{\;i}_{kh} + \sim_{\ell m} H^{\;i}_{jkh} .$$

we now use the commutation formula (1.2) and get

(2.46) $$\left\{\dot{\partial}_{j}\left(H^{\;i}_{kh|m}\right)\right\}_{|\ell} + H^{\;r}_{kh|m} \dot{\partial}_{j} \Gamma^{*i}_{r\ell} - H^{\;i}_{rh|m} \dot{\partial}_{j} \Gamma^{*r}_{k\ell} - H^{\;i}_{kr|m} \dot{\partial}_{j} \Gamma^{*r}_{h\ell} \\ - H^{\;i}_{kh|r} \dot{\partial}_{j} \Gamma^{*r}_{m\ell} - \dot{\partial}_{r}\left(H^{\;i}_{kh|m}\right) P^{r}_{j\ell} = \left(\dot{\partial}_{j} \text{a}_{\ell}\right) H^{\;i}_{kh|m} + \text{a}_{\ell} H^{\;i}_{jkh|m} + \\ + \text{a}_{\ell}\left(H^{\;r}_{kh} \dot{\partial}_{j} \Gamma^{*i}_{rm}\right) - H^{\;i}_{rh} \dot{\partial}_{j} \Gamma^{*r}_{km} - H^{\;i}_{kr} \dot{\partial}_{j} \Gamma^{*r}_{hm} - H^{\;i}_{rkh} P^{r}_{jm}\Big) + \\ + \left(\dot{\partial}_{j} \sim_{\ell m}\right) H^{\;i}_{kh} + \sim_{\ell m} H^{\;i}_{jkh} .$$

Using the commutation formula (1.2) in (2.46), we get

(2.47) $$H^{\;i}_{jkh|m|\ell} + \Big\{\Big(H^{\;r}_{kh} \dot{\partial}_{j} \Gamma^{*i}_{rm} - H^{\;i}_{rh} \dot{\partial}_{j} \Gamma^{*r}_{km} - H^{\;i}_{kr} \dot{\partial}_{j} \Gamma^{*r}_{hm} - \\ - H^{\;i}_{rkh} P^{r}_{jm}\Big)_{|\ell} + H^{\;r}_{kh|m} \dot{\partial}_{j} \Gamma^{*i}_{r\ell} - H^{\;i}_{rh|m} \dot{\partial}_{j} \Gamma^{*r}_{k\ell} - H^{\;i}_{kr|m} \dot{\partial}_{j} \Gamma^{*r}_{h\ell} - \\ - H^{\;i}_{kh|r} \dot{\partial}_{j} \Gamma^{*r}_{m\ell} - H^{\;i}_{rkh|m} P^{r}_{j\ell} - H^{\;s}_{kh} \dot{\partial}_{r} \Gamma^{*i}_{sm} P^{r}_{j\ell} + H^{\;i}_{sh} \dot{\partial}_{r} \Gamma^{*s}_{km} P^{r}_{j\ell} +$$

IJMDRR
E- ISSN –2395-1885
ISSN -2395-1877

$$+H^{\,i}_{ks}\dot{\partial}_r\Gamma^{*s}_{hm}P^{\,r}_{j\ell}+H^{\,i}_{skh}P^{\,s}_{rm}P^{\,r}_{j\ell}\Big\}$$
$$=\left(\mathsf{a}_{\ell}H^{\,i}_{jkh|m}+\sim_{\ell m}H^{\,i}_{jkh}\right)+\Big\{\left(\dot{\partial}_j\mathsf{a}_{\ell}\right)H^{\,i}_{kh|m}+\mathsf{a}_{\ell}H^{\,r}_{kh}\dot{\partial}_j\Gamma^{*i}_{rm}-$$
$$-\mathsf{a}_{\ell}H^{\,i}_{rh}\dot{\partial}_j\Gamma^{*r}_{km}-\mathsf{a}_{\ell}H^{\,i}_{kr}\dot{\partial}_j\Gamma^{*r}_{hm}-\mathsf{a}_{\ell}H^{\,i}_{rhk}P^{\,r}_{j\ell m}+\left(\dot{\partial}_j\sim_{\ell m}\right)H^{\,i}_{kh}\Big\}.$$

From (2.47) it is obvious that

(2.48) $H^{\,i}_{jkh|m|\ell}=\mathsf{a}_{\ell}H^{\,i}_{jkh|m}+\sim_{\ell m}H^{\,i}_{jkh}$,

If and only if

(2.49) $$\left(H^{\,r}_{kh}\dot{\partial}_j\Gamma^{*i}_{rm}-H^{\,i}_{rh}\dot{\partial}_j\Gamma^{*r}_{km}-H^{\,i}_{kr}\dot{\partial}_j\Gamma^{*r}_{hm}-H^{\,i}_{rkh}P^{\,r}_{jm}\right)_{|\ell}+$$
$$+H^{\,r}_{kh|m}\Gamma^{*i}_{r\ell}-H^{\,i}_{rh|m}\dot{\partial}_j\Gamma^{*r}_{k\ell}-H^{\,i}_{kr|m}\dot{\partial}_j\Gamma^{*r}_{h\ell}-H^{\,i}_{kh|r}\dot{\partial}_j\Gamma^{*r}_{m\ell}-$$
$$-H^{\,i}_{rkh|m}P^{\,r}_{jk}-H^{\,s}_{kh}\dot{\partial}_r\Gamma^{*i}_{sm}P^{\,r}_{j\ell}+H^{\,i}_{sh}\dot{\partial}_r\Gamma^{*s}_{km}P^{\,r}_{j\ell}+$$
$$+H^{\,i}_{ks}\dot{\partial}_r\Gamma^{*s}_{hm}P^{\,r}_{j\ell}=\left(\dot{\partial}_j\mathsf{a}_{\ell}\right)H^{\,i}_{kh|m}+$$
$$+\mathsf{a}_{\ell}H^{\,r}_{kh}\dot{\partial}_j\Gamma^{*i}_{rm}-\mathsf{a}_{\ell}H^{\,i}_{rh}\dot{\partial}_j\Gamma^{*r}_{km}-\mathsf{a}_{\ell}H^{\,i}_{kr}\dot{\partial}_j\Gamma^{*r}_{hm}-\mathsf{a}_{\ell}H^{\,i}_{rkh}P^{\,r}_{jm}+$$
$$+\left(\dot{\partial}_j\sim_{\ell m}\right)H^{\,i}_{kh}.$$

Therefore we can state:

Theorem (2.7)

The Berwald's curvature tensor $H^{\,i}_{jkh}$ of an R^h–generalised birecurrent space of the first kind is h -generalised birecurrent of the first kind if and only if (2.49) holds.

After adopting the process similar to that which have been studied in the foregoing lines for (2.34), (2.35) and (2.36), we may state the following:

Theorem (2.8)

The Berwald's curvature tensor $H^{\,i}_{jkh}$ of an R^h–generalised birecurrent Finsler space of the second kind is h -generalised birecurrent of the second kind if and only if

(2.50) $$\left(H^{\,r}_{kh}\dot{\partial}_j\Gamma^{*i}_{rm}-H^{\,i}_{rh}\dot{\partial}_j\Gamma^{*r}_{km}-H^{\,i}_{kr}\dot{\partial}_j\Gamma^{*r}_{hm}-H^{\,i}_{rkh}P^{\,r}_{jm}\right)_{|\ell}+$$
$$+H^{\,r}_{kh|m}\dot{\partial}_j\Gamma^{*i}_{r\ell}-H^{\,i}_{rh|m}\dot{\partial}_j\Gamma^{*r}_{k\ell}-H^{\,i}_{kr|m}\dot{\partial}_j\Gamma^{*r}_{h\ell}-H^{\,i}_{kh|r}\dot{\partial}_j\Gamma^{*r}_{m\ell}+$$
$$+H^{\,i}_{rkh|m}P^{\,r}_{j\ell}=\left(\dot{\partial}_j\mathsf{a}_{m}\right)H^{\,i}_{kh|\ell}+\mathsf{a}_{m}H^{\,r}_{kh}\dot{\partial}_j\Gamma^{*i}_{r\ell}-\mathsf{a}_{m}H^{\,i}_{rh}\dot{\partial}_j\Gamma^{*r}_{k\ell}-$$
$$-\mathsf{a}_{m}H^{\,i}_{kr}\dot{\partial}_j\Gamma^{*r}_{h\ell}-\mathsf{a}_{m}H^{\,i}_{rkh}P^{\,r}_{j\ell}+\left(\dot{\partial}_j\sim_{\ell m}\right)H^{\,i}_{kh}$$

holds.

Theorem (2.9)

The Berwald's curvature tensor $H^{\,i}_{jkh}$ of an R^h – special generalised birecurrent Finsler space of the first kind is h -special generalised birecurrent of the first kind if and only if

(2.51) $$\left(H^{\,r}_{kh}\dot{\partial}_j\Gamma^{*i}_{rm}-H^{\,i}_{rh}\dot{\partial}_j\Gamma^{*r}_{km}-H^{\,i}_{kr}\dot{\partial}_j\Gamma^{*r}_{hm}-H^{\,i}_{rkh}P^{\,r}_{jm}\right)_{|\ell}+$$
$$+H^{\,r}_{kh|m}\dot{\partial}_j\Gamma^{*i}_{r\ell}-H^{\,i}_{rh|m}\dot{\partial}_j\Gamma^{*r}_{k\ell}-H^{\,i}_{kr|m}\dot{\partial}_j\Gamma^{*r}_{h\ell}-H^{\,i}_{kh|r}\dot{\partial}_j\Gamma^{*r}_{m\ell}-$$
$$-H^{\,i}_{rkh|m}P^{\,r}_{j\ell}-H^{\,s}_{kh}\dot{\partial}_r\Gamma^{*i}_{sm}P^{\,r}_{j\ell}+H^{\,i}_{sh}\dot{\partial}_r\Gamma^{*s}_{km}P^{\,r}_{j\ell}+H^{\,i}_{ks}\dot{\partial}_r\Gamma^{*s}_{hm}P^{\,r}_{j\ell}+$$
$$+H^{\,i}_{skh|m}P^{\,s}_{rm}P^{\,r}_{j\ell}=\left(\dot{\partial}_j\mathsf{a}_{\ell}\right)H^{\,i}_{kh|m}+\mathsf{a}_{\ell}H^{\,r}_{kh}\dot{\partial}_j\Gamma^{*i}_{rm}-\mathsf{a}_{\ell}H^{\,i}_{rh}\dot{\partial}_j\Gamma^{*r}_{km}-$$

$$-\lambda_{\ell} H^{i}_{kr}\dot{\partial}_{j}\Gamma^{*r}_{hm} - \lambda_{\ell} H^{i}_{rkh} P^{r}_{jm}$$

holds.

Theorem (2.10)

The Berwald's curvature tensor H^{i}_{jkh} of an R^{h}–generalised birecurrent Finsler space of the second kind is h- generalised birecurrent of the second kind if and only if

(2.52) $$\left(H^{r}_{kh}\dot{\partial}_{j}\Gamma^{*i}_{rm} - H^{i}_{rh}\dot{\partial}_{j}\Gamma^{*r}_{km} - H^{i}_{kr}\dot{\partial}_{j}\Gamma^{*r}_{km} - H^{i}_{rkh}P^{r}_{jm}\right)_{|\ell} +$$

$$+ H^{r}_{kh|m}\dot{\partial}_{j}\Gamma^{*i}_{r\ell} - H^{i}_{rh|m}\dot{\partial}_{j}\Gamma^{*r}_{k\ell} - H^{i}_{kr|m}\dot{\partial}_{j}\Gamma^{*r}_{h\ell} - H^{i}_{kh|r}\dot{\partial}_{j}\Gamma^{*r}_{m\ell} -$$

$$- H^{i}_{rkh|m}P^{r}_{j\ell} - H^{s}_{kh}\dot{\partial}_{r}\Gamma^{*i}_{sm}P^{r}_{j\ell} + H^{i}_{sh}\dot{\partial}_{r}\Gamma^{*s}_{km}P^{r}_{j\ell} +$$

$$+ H^{i}_{ks}\dot{\partial}_{r}\Gamma^{*s}_{hm}P^{r}_{j\ell} + H^{i}_{skh}P^{s}_{rm}P^{r}_{j\ell} = \left(\dot{\partial}_{j}\lambda_{m}\right)H^{i}_{kh|\ell} +$$

$$+ \lambda_{m} H^{r}_{kh}\dot{\partial}_{j}\Gamma^{*i}_{rl} - \lambda_{m} H^{i}_{rh}\dot{\partial}_{j}\Gamma^{h}_{k\ell} - \lambda_{m} H^{i}_{kr}\dot{\partial}_{j}\Gamma^{*r}_{h\ell} - \lambda_{m} H^{i}_{rkh}P^{r}_{j\ell}$$

Holds.

The Identites of A R^{h}–Generalised And R^{h}–Special Generalised Birecurrent Finsler Space of The Two Kinds

The identity satisfied by the curvature tensor R_{ijkh} has been given by Cartan [2] as

(3.1) $$R_{ijkh} + R_{ihkj} + R_{ikjh} + \left(C_{ijs}K^{s}_{rhk} + C_{ihs}K^{s}_{rkj} + C_{iks}K^{s}_{rjh}\right)\dot{x}^{r} = 0.$$

Using (1.9) in (3.1), we get

(3.2) $$R_{ijkh} + R_{ihkj} + R_{ikjh} + C_{ijs}H^{s}_{hk} + C_{ihs}H^{s}_{kj} + C_{iks}H^{s}_{jh} = 0.$$

Differentiating (3.2) covariantly with respect to x^{m} in the sense of Cartan, we get

(3.3) $$R_{ijkh|m} + R_{ihkj|m} + R_{ikjh|m} +$$

$$+\left(C_{ijs}H^{s}_{hk} + C_{ihs}H^{s}_{kj} + C_{iks}H^{s}_{jh}\right)_{|m} = 0$$

Differentiating (3.3) covariantly with respect to x^{ℓ} in the sense of Cartan and using (2.7), (2.8), (2.9) and (2.10) separately, we get the following

(3.4) $$\lambda_{\ell}\left(R_{ijhk|m} + R_{ihkj|m} + R_{ikjh|m}\right) + \mu_{\ell m}\left(R_{ijhk} + R_{ihkj} + R_{ikjh}\right) +$$

$$\left(C_{ijs}H^{s}_{hk} + C_{ihs}H^{s}_{kj} + C_{iks}H^{s}_{jh}\right)_{|m|\ell} = 0$$

(3.5) $$\lambda_{m}\left(R_{ijhk|m} + R_{ihkj|m} + R_{ikjh|m}\right) + \mu_{\ell m}\left(R_{ijhk} + R_{ihkj} + R_{ikjh}\right) +$$

$$+\left(C_{ijs}H^{s}_{hk} + C_{ihs}H^{s}_{kj} + C_{iks}H^{s}_{jh}\right)_{|m|\ell} = 0$$

(3.6) $$\lambda_{\ell}\left(R_{ijhk|m} + R_{ihkj|m} + R_{ikjh|m}\right) +$$

$$+\left(C_{ijs}H^{s}_{hk} + C_{ihs}H^{s}_{kj} + C_{iks}H^{s}_{jh}\right)_{|m|\ell} = 0,$$

(3.7) $$\lambda_{m}\left(R_{ijhk|\ell} + R_{ihkj|\ell} + R_{ikjh|\ell}\right) +$$

$$+\left(C_{ijs}H^{s}_{hk} + C_{ihs}H^{s}_{kj} + C_{iks}H^{s}_{jh}\right)_{|m|\ell} = 0,$$

We now use (3.2) and (3.3) in (3.4), (3.5),(3.6),(3.7) respectively and get

(3.8) $$\left(C_{ijs}H^{s}_{hk} + C_{ihs}H^{s}_{kj} + C_{iks}H^{s}_{jh}\right)_{|m|\ell}$$

IJMDRR
E- ISSN –2395-1885
ISSN -2395-1877

$$= \text{a}_{\ell}\left(C_{ijs}H^{s}_{hk} + C_{ihs}H^{s}_{kj} + C_{iks}H^{s}_{jh}\right)_{|m} +$$
$$+\sim_{\ell m}\left(C_{ijs}H^{s}_{hk} + C_{ihs}H^{s}_{kj} + C_{iks}H^{s}_{jh}\right),$$

(3.9) $$\left(C_{ijs}H^{s}_{hk} + C_{ihs}H^{s}_{kj} + C_{iks}H^{s}_{jh}\right)_{|m|\ell}$$
$$= \text{a}_{m}\left(C_{ijs}H^{s}_{hk} + C_{ihs}H^{s}_{kj} + C_{iks}H^{s}_{jh}\right)_{|\ell} +$$
$$+\sim_{\ell m}\left(C_{ijs}H^{s}_{hk} + C_{ihs}H^{s}_{kj} + C_{iks}H^{s}_{jh}\right),$$

(3.10) $$\left(C_{ijs}H^{s}_{hk} + C_{ihs}H^{s}_{kj} + C_{iks}H^{s}_{jh}\right)_{|m|\ell}$$
$$= \text{a}_{\ell}\left(C_{ijs}H^{s}_{hk} + C_{ihs}H^{s}_{kj} + C_{iks}H^{s}_{jh}\right)_{|m}$$

(3.11) $$\left(C_{ijs}H^{s}_{hk} + C_{ihs}H^{s}_{kj} + C_{iks}H^{s}_{jh}\right)_{|m|\ell}$$
$$= \text{a}_{m}\left(C_{ijs}H^{s}_{hk} + C_{ihs}H^{s}_{kj} + C_{iks}H^{s}_{jh}\right)_{|\ell}.$$

Transecting (3.8),(3.9), (3.10) and (3.11) by $\dot{x}^{j}$ and using (1.7) thereafter, we get

(3.12) $$\left(C_{iks}H^{s}_{h} - C_{ihs}H^{s}_{k}\right)_{|m|\ell} = \text{a}_{\ell}\left(C_{iks}H^{s}_{h} - C_{ihs}H^{s}_{k}\right)_{|m} +$$
$$+\sim_{\ell m}\left(C_{iks}H^{s}_{h} - C_{ihs}H^{s}_{k}\right),$$

(3.13) $$\left(C_{iks}H^{s}_{h} - C_{ihs}H^{s}_{k}\right)_{|m|\ell} = \text{a}_{m}\left(C_{iks}H^{s}_{h} - C_{ihs}H^{s}_{k}\right)_{|\ell} +$$
$$+\sim_{\ell m}\left(C_{iks}H^{s}_{h} - C_{ihs}H^{s}_{k}\right)$$

(3.14) $$\left(C_{iks}H^{s}_{h} - C_{ihs}H^{s}_{k}\right)_{|m|\ell} = \text{a}_{m}\left(C_{iks}H^{s}_{h} - C_{ihs}H^{s}_{k}\right)_{|\ell},$$

(3.15) $$\left(C_{iks}H^{s}_{h} - C_{ihs}H^{s}_{k}\right)_{|m|\ell} = \text{a}_{\ell}\left(C_{iks}H^{s}_{h} - C_{ihs}H^{s}_{k}\right)_{|m}.$$

Transvecting (3.8),(3.9), (3.10) and (3.11) successively by g^{pi} and thereafter using the symmetry property of the tensor C_{ijk} in all its lower indices along with the fact that
$C_{ijk}\dot{x}^{i} = C_{ijk}\dot{x}^{j} = C_{ijk}\dot{x}^{k} = 0$, we get

(3.16) $$\left(C^{p}_{ks}H^{s}_{h} - C^{p}_{hs}H^{s}_{k}\right)_{|m|\ell} = \text{a}_{\ell}\left(C^{p}_{ks}H^{s}_{h} - C^{p}_{hs}H^{s}_{k}\right)_{|m} +$$
$$+\sim_{\ell m}\left(C^{p}_{ks}H^{s}_{h} - C^{p}_{hs}H^{s}_{k}\right),$$

(3.17) $$\left(C^{p}_{ks}H^{s}_{h} - C^{p}_{hs}H^{s}_{k}\right)_{|m|\ell} = \text{a}_{m}\left(C^{p}_{ks}H^{s}_{h} - C^{p}_{hs}H^{s}_{k}\right)_{|\ell} +$$
$$+\sim_{\ell m}\left(C^{p}_{ks}H^{s}_{h} - C^{p}_{hs}H^{s}_{k}\right),$$

(3.18) $$\left(C^{p}_{ks}H^{s}_{h} - C^{p}_{hs}H^{s}_{k}\right)_{|m|\ell} = \text{a}_{\ell}\left(C^{p}_{ks}H^{s}_{h} - C^{p}_{hs}H^{s}_{k}\right)_{|m},$$

(3.19) $$\left(C^{p}_{ks}H^{s}_{h} - C^{p}_{hs}H^{s}_{k}\right)_{|m|\ell} = \text{a}_{m}\left(C^{p}_{ks}H^{s}_{h} - C^{p}_{hs}H^{s}_{k}\right)_{|\ell}.$$

In view of (1.4) and (1.5), the equation (3.3) can be rewritten as

(3.20) $$R^{i}_{jhk|m} + R^{i}_{hkj|m} + R^{i}_{kjh|m} + \left(C^{i}_{js}H^{s}_{hk} + C^{i}_{hs}H^{s}_{kj} + C^{i}_{ks}H^{s}_{jh}\right) = 0.$$

We now use (1.9) in (3.17), we get

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E- ISSN –2395-1885
ISSN -2395-1877

(3.21) $a_m \left(K^i_{jhk} + K^i_{hkj} + K^i_{kjh} \right) + 2\left(C^i_{js} H^s_{hk} + C^i_{hs} H^s_{kj} + C^i_{ks} H^s_{jh} \right) = 0$

Using (1.4) in (3.17), we get

(3.22) $\left(C^i_{js} H^s_{hk} + C^i_{hs} H^s_{kj} + C^i_{ks} H^s_{jh} \right) = 0$

(3.18) obviously implies

(3.23) $C^i_{js|m} H^s_{hk} + C^i_{js} H^s_{hk|m} + C^i_{hs|m} H^s_{kj} + C^i_{hs} H^s_{kj|m} +$

$$+ C^i_{ks|m} H^s_{jh} + C^i_{ks} H^s_{jh|m} = 0.$$

Differentiating (3.19) with respect to x^ℓ in the sense of Cartan, we get

(3.24) $C^i_{js|m|\ell} H^s_{hk} + C^i_{js|m} H^s_{hk|\ell} + C^i_{js|\ell} H^s_{hk|m} + C^i_{js} H^s_{hk|m|\ell} +$

$$+ C^i_{hs|m|\ell} H^s_{kj} + C^i_{hs|m} H^s_{kj|\ell} + C^i_{hs|\ell} H^s_{kj|m} + C^i_{hs} H^s_{kj|m|\ell} +$$

$$+ C^i_{ks|m} H^s_{jh|\ell} + C^i_{ks|m|\ell} H^s_{jh} + C^i_{ks} H^s_{jh|m|\ell} + C^i_{ks|\ell} H^s_{jh|m} = 0.$$

Transvecting (3.20) by $\dot{x}^m$ and therefore using (1.5), we get

(3.25) $P^i_{js|\ell} H^s_{hk} + P^i_{js} H^s_{hk|\ell} + C^i_{js|\ell} H^s_{hk|m} \dot{x}^m + C^i_{js} H^s_{hk|m|\ell} \dot{x}^m +$

$$P^i_{hs|\ell} H^s_{kj} + P^i_{hs} H^s_{kj|\ell} + C^i_{hs|\ell} H^s_{kj|m} \dot{x}^m + C^i_{hs} H^s_{kj|m|\ell} \dot{x}^m +$$

$$P^i_{ks|\ell} H^s_{jk} + P^i_{ks} H^s_{jh|\ell} + C^i_{ks|\ell} H^s_{jh|m} \dot{x}^m + C^i_{ks} H^s_{jh|m|\ell} \dot{x}^m = 0$$

Transvecting (3.20) by $\dot{x}^h$ and using (1.4) and the fact that $P^i_{kh} \dot{x}^h = 0$, we get

(3.26) $\left(P^i_{js} H^s_k \right)_{|\ell} - \left(P^i_{ks} H^s_j \right)_{|\ell} + \left(C^i_{js} H^s_{k|m} - C^i_{ks} H^s_{h|m} \right)_{|\ell} \dot{x}^m = 0.$

Differentiating the identity $R^r_{ijk|h} + R^r_{ihj|k} + R^r_{ikh|j} + \dot{x}^m \left(R^r_{mkh} P^r_{ij\ell} + R^\ell_{mjk} P^r_{ih\ell} + R^\ell_{mnj} P^r_{ih\ell} \right) = 0$, with respect to x^ℓ in sense of Cartan, we get

(3.27) $R^i_{jkh|m|\ell} + R^i_{jmk|h|\ell} + R^i_{jhm|k|\ell} +$

$$+ \dot{x}^s \left(R^s_{rhm} P^i_{jks} + R^s_{rkh} P^i_{jms} + R^s_{rmk} P^i_{jhs} \right)_{|\ell} = 0 .$$

In view of the conditions as have been given in (2.3), (2.4), (2.5) and (2.6), (3.23) to (3.26) may be written as

(3.28) $a_\ell \left(R^i_{jkh|m} + R^i_{jmk|h} + R^i_{jhm|k} \right) +$

$$+ \sim_{\ell m} R^i_{jkh} + \sim_{\ell h} R^i_{jmk} + \sim_{\ell k} R^i_{jhm} +$$

$$+\left(H^{r}_{hm}P^{i}_{jkr}+H^{r}_{kh}P^{i}_{jmr}+H^{r}_{mk}P^{i}_{jhr}\right)_{|\ell}=0\,,$$

(3.29) $$\mathrm{a}_{m}R^{i}_{jkh|\ell}+\sim_{\ell m}R^{i}_{jkh}+\mathrm{a}_{h}R^{i}_{jmk|\ell}+\sim_{\ell h}R^{i}_{jmk}+\mathrm{a}_{k}R^{i}_{jhm|\ell}+$$

$$\sim_{\ell k}R^{i}_{jhm}+\left(H^{r}_{hm}P^{i}_{jkr}+H^{r}_{kh}P^{i}_{jmr}+H^{r}_{mk}P^{i}_{jhr}\right)_{|\ell}=0\,,$$

(3.30) $$\mathrm{a}_{\ell}\left(R^{i}_{jkh|m}+R^{i}_{jmk|h}+R^{i}_{jhm|k}\right)+$$

$$+\left(H^{r}_{hm}P^{i}_{jkr}+H^{r}_{kh}P^{i}_{jmr}+H^{r}_{mk}P^{i}_{jhr}\right)_{|\ell}=0\,,$$

And

(3.31) $$\left(\mathrm{a}_{m}R^{i}_{jk|\ell}+\mathrm{a}_{h}R^{i}_{jmk|\ell}+\mathrm{a}_{k}R^{i}_{jhm|\ell}\right)+$$

$$+\left(H^{r}_{hm}P^{i}_{jkr}+H^{r}_{kh}P^{i}_{jmr}+H^{r}_{mk}P^{i}_{jhr}\right)_{|\ell}=0\,.$$

Transvecting (3.24), (3.25), (3.26) and (3.27) by $\dot{x}^{j}$ and thereafter using (1.4) and (1.5) we get,

(3.32) $$\mathrm{a}_{\ell}\left(H^{i}_{kh|m}+H^{i}_{mk|h}+H^{i}_{hm|k}\right)+$$

$$+\left(\sim_{\ell m}H^{i}_{kh}+\sim_{\ell h}H^{i}_{mk}+\sim_{\ell k}H^{i}_{hm}\right)+$$

$$+\left(H^{r}_{hm}P^{i}_{kr}+H^{r}_{kh}P^{i}_{mr}+H^{r}_{mk}P^{i}_{hr}\right)_{|\ell}=0\,,$$

(3.33) $$\mathrm{a}_{\ell}\left(H^{i}_{kh|m}+H^{i}_{mk|h}+H^{i}_{hm|k}\right)+$$

$$+\left(H^{r}_{hm}P^{i}_{kr}+H^{r}_{kh}P^{i}_{mr}+H^{r}_{mk}P^{i}_{hr}\right)_{|\ell}=0$$

And

(3.34) $$\left(\mathrm{a}_{m}H^{i}_{kh|\ell}+\mathrm{a}_{h}H^{i}_{mk|\ell}+\mathrm{a}_{k}H^{i}_{hm|\ell}\right)+$$

$$+\left(H^{r}_{hm}P^{i}_{kr}H^{r}_{kh}P^{i}_{mr}+H^{r}_{mk}P^{i}_{hr}\right)_{|\ell}=0\,.$$

Therefore, we may state the following:

Theorem (3.1)

In an R^h -generalised birecurrent Finsler space of the first kind the identities (3.8), (3.9), (3.12), (3.13), (3.16) and (3.17) hold and the tensors $C_{ijs}H^s_{hk} + C_{ihs}H^s_{kj} + C_{iks}H^s_{jh}$, $C_{iks}H^s_h - C_{ihs}H^s_k$ and $C^p_{ks}H^s_h - C^p_{hs}H^s_k$ are all h -generalized birecurrent of the first kind.

Theorem (3.2)

In an R^h -generalised birecurrent Finsler space of the second kind the identities (3.8), (3.9), (3.12), (3.13), (3.16) and (3.17) hold and the tensors $C_{ijk}H^s_{hk} + C_{ihk}H^s_{kj} + C_{iks}H^s_{jh}$, $C_{iks}H^s_h - C_{ihs}H^s_k$ and $C^p_{ks}H^s_h - C^p_{hs}H^s_k$ are all h -generalised birecurrent of the second kind.

Theorem (3.3)

In an R^h -special generalised birecurrent Finsler space of the first kind the identities (3.24), (3.26) and (3.33) hold and the tensors $C_{ijs}H^s_{hk} + C_{ihs}H^s_{hk} + C_{iks}H^s_{jh}$, $C_{iks}H^s_h - C_{ihs}H^s_k$ and $C^p_{ks}H^s_h - C^p_{hs}H^s_k$ are all h -special generalised birecurrent of the first kind.

Theorem (3.4)

In an R^h -special generalised birecurrent Finsler space of the second kind the identities (3.24), (3.26) and (3.34) hold and the tensors $C_{ijs}H^s_{hk} + C_{ihs}H^s_{kj} + C_{iks}H^s_{jh}$, $C_{iks}H^s_h - C_{ihs}H^s_k$ and $C^p_{ks}H^s_h - C^p_{hs}H^s_k$ are all h -special generalised birecurrent of the second kind.

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